

Computer Science Department

TECHNICAL REPORT

ITERATIVE METHODS FOR ELLIPTIC PROBLEMS ON REGION
PARTITIONED INTO SUBSTRUCTURES AND THE
BIHARMONIC DIRICHLET PROBLEM

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Olof B. Widlund

January 1984

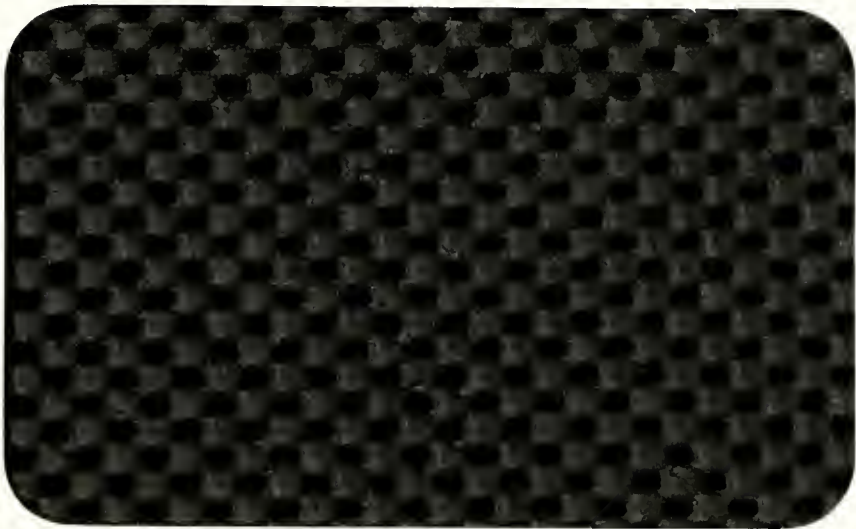
Report #101

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Iterative Methods for Elliptic Problems on Regions
Partitioned into Substructures and the
Biharmonic Dirichlet Problem

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*Prepared for the Proceedings of the Sixth International Conference on
Computing Methods in Applied Science and Engineering, held at Versailles,
France, December 12-16, 1983.

ITERATIVE METHODS FOR ELLIPTIC PROBLEMS ON REGIONS PARTITIONED INTO SUBSTRUCTURES AND THE BIHARMONIC DIRICHLET PROBLEM

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A finite element problem can often naturally be partitioned into subproblems which correspond to subregions into which the region has been partitioned or from which it was originally assembled. A class of methods are discussed in which these subproblems are solved by direct methods while the interaction between the subregions is handled by a conjugate gradient algorithm. The mathematical framework for this work is provided by regularity theory for elliptic finite element equations and block-Gaussian elimination. The same tools can be used to provide a reinterpretation of Björstad's work on very fast methods for the biharmonic Dirichlet problem on rectangles and a variant of an algorithm introduced by Glowinski and Pironneau for mixed finite element approximations of the same elliptic problem on general regions. The relationship between substructuring and finite element-capacitance matrix techniques is also briefly explored.

INTRODUCTION

In this paper, we will discuss some special iterative methods for elliptic finite element problems defined on regions regarded as unions of subregions. We are interested, in particular, in the design of algorithms for which the interaction between the subproblems, i.e. the discrete elliptic problems on the subregions, is computed with the aid of a conjugate gradient method, while the subproblems themselves are solved by a direct method. This approach differs from the standard one used in industry where direct methods are employed throughout.

The partition of elliptic problems into subproblems is a very natural idea that is widely used in practice and which has been discussed in the engineering literature for at least twenty years; see e.g. Prezemieniecki [18,19]. Thus the modeling of an entire mechanical system can profitably be organized by discretizing the partial differential equations by finite elements on subregions. These tasks and the factorization of the resulting stiffness matrices for the subproblems can be assigned to different engineering groups and computer systems or processors with coordination required only at the interfaces between the substructures to assure matching of finite element triangulations and solutions. These ideas are particularly attractive for parallel computing and in the case when some of the substructures are identical or previously analyzed as in the case when a simulation is repeated after the redesign or damage of one or a few of the substructures. The final phase, in which the interaction is accounted for, does not lend itself equally well to parallel computing and it also represents a significant fraction of the total computational work even on a sequential computer.

Three alternative methods are discussed in this paper and numerical results for a model problem are presented. These algorithms are analyzed in detail in a forth-

coming paper by Bjørstad and Widlund [4], in which a complete theory for conforming Lagrangian finite element approximations of second order elliptic problems in the plane are given. The development of this theory requires the use of elliptic regularity theory for problems on Lipschitz regions; see Grisvard [15], and some apparently new finite element results. We regard the extension of these results to more general elliptic problems as a significant open problem.

Earlier work on algorithms of this kind is reported in Concus, Golub and O'Leary [6], who gave numerical results for Poisson's problem on T-shaped regions. For an analysis of this method see Bjørstad and Widlund [4]. Two interesting new preconditioners for the same special problem are introduced in the contribution to these proceedings by Golub and Mayers [12]. In their numerical experiments a comparison is made between these algorithms and the one characterized as second best in our previous paper; cf. Bjørstad and Widlund [3]. Methods inspired by Schwarz' alternating method and by control theory are considered in Glowinski, Periaux and Dihn [10]. For a discussion of one of these, see Bjørstad and Widlund [4]. There has also been extensive activity in the Soviet Union in this area. We are not yet well acquainted with this work and are grateful to Yu. Kuznetsov and Maximilian Dryja for drawing our attention to it. After our first series of numerical experiments, we received a preprint of one of Dryja's papers [8], from which we first learned of what we currently regard as the best algorithm. In that paper Dryja analyzes the case of Laplace's equation on L-shaped regions.

The techniques developed for the substructuring problems, which combine block Gaussian elimination and regularity theorems for elliptic problems, are also useful in examining certain iterative methods for the biharmonic problem. Even on rectangular regions, this problem is of considerable difficulty, since separation of variables cannot be used directly. By introducing an auxiliary variable, known as vorticity in fluid dynamics, the problem can be reformulated as a second order elliptic system. Many iterative methods have been developed to solve this system in which a different, simpler boundary value problem is solved in each step; see Bjørstad [1,2] and Glowinski and Pironneau [11] and the many references given in those papers. In this paper, we introduce an alternative to an optimal method given by Glowinski and Pironneau [11] and give a reinterpretation of Bjørstad's work.

While for substructured problems we seek to solve a large linear system by an iterative method which involves smaller problems, there are occasions when we are willing to solve a larger linear system repeatedly, to obtain the solution of a smaller system. Such algorithms are known as capacitance matrix methods; see O'Leary and Widlund [17], Proskurowski and Widlund [20,21] for a discussion and references to the literature. In our last section, we explore the relationship between substructuring and finite element-capacitance matrix methods.

SUBSTRUCTURED FINITE ELEMENT PROBLEMS AND BLOCK FORM OF THE STIFFNESS MATRICES

To simplify the discussion, we confine ourselves to problems defined on the union Ω of Ω_1 , Ω_2 and Γ_3 and to the Dirichlet case. Here Ω_1 and Ω_2 are plane, bounded, nonintersecting regions and Γ_3 the intersection of their closures. The boundaries of Ω_1 and Ω_2 are $\Gamma_1 \cup \Gamma_3$ and $\Gamma_2 \cup \Gamma_3$, respectively, and the boundary of Ω is $\Gamma_1 \cup \Gamma_2$. We assume that these subregions are curvilinear polygons, i.e. in particular they are Lipschitz regions; see Grisvard [13]. A linear, second order, positive definite, selfadjoint elliptic operator is defined on Ω . Its symmetric bilinear form is denoted by $a_\Omega(u,v)$; see Ciarlet [5] or Strang and Fix [22]. A simple example is given by the Laplace operator with homogeneous Dirichlet condition for which

$$a_{\Omega}(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx, \quad u, v \in H_0^1(\Omega).$$

Here $H_0^1(\Omega)$ is the subspace of elements, with zero boundary values, of the Sobolev space of square integrable functions with square integrable first derivatives. Triangulations of Ω_1 and Ω_2 are introduced in such a way that the nodes on Γ_3 coincide and Γ_3 follows element boundaries. We assume that each degree of freedom of the finite element subspaces is associated with a node and a basis function in the finite element space and that the support of any basis function coincides with the triangles to which its node belongs. An element of the stiffness matrix has the form $a_{\Omega}(\phi_i, \phi_j)$ where ϕ_i and ϕ_j are basis functions, and it therefore vanishes unless the two nodes belong to a common triangle.

It is easy to see from the definition of the bilinear form that

$$a_{\Omega_1}(u, v) + a_{\Omega_2}(u, v) = a_{\Omega}(u, v) \quad (2.1)$$

and that therefore the stiffness matrix of a problem on Ω can be constructed from those of Ω_1 and Ω_2 . The same relation holds for any pair of nonoverlapping subregions. This fact is frequently used in practice to construct stiffness matrices from the stiffness matrices of the individual triangles; see Strang and Fix [22].

We write the stiffness matrices corresponding to Ω_1 and Ω_2 respectively as

$$K^{(1)} = \begin{pmatrix} K_{11} & K_{13} \\ K_{13}^T & K_{33} \end{pmatrix} \quad \text{and} \quad K^{(2)} = \begin{pmatrix} K_{22} & K_{23} \\ K_{23}^T & K_{33} \end{pmatrix}, \quad (2.2)$$

where K_{11} represents couplings between the pairs of nodes in Ω_1 , K_{13} couplings between the pairs belonging to Ω_1 and Γ_3 respectively and $K_{33}^{(1)}$ couplings between nodes on Γ_3 , etc.

Written in variational form a Dirichlet problem on Ω_1 has the form

$$\begin{aligned} \int_{\Omega_1} \nabla u_h \cdot \nabla v_h \, dx &= \int_{\Omega_1} f v_h \, dx, \quad \forall v_h \in S^h(\Omega_1) \cap H_0^1(\Omega_1), \\ u_h &\in S^h(\Omega_1), \quad \gamma_0 u_h = g_0, \end{aligned}$$

where $S^h(\Omega_1) \subset H^1(\Omega)$ is a finite element subspace and $\gamma_0 u_h$ is the trace, i.e. the restriction of u_h to the boundary. In matrix form this problem can be written as

$$\begin{pmatrix} K_{11} & K_{13} \\ 0 & I \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_3 \end{pmatrix} \quad (2.3)$$

where x_1 and x_3 are the vectors of nodal values corresponding to the open set Ω_1 and Γ_3 respectively. The components of b_1 are constructed from the right hand side f and the Dirichlet values on Γ_1 and b_3 is the vector of Dirichlet

values on Γ_3 .

If on the other hand Neumann data g_N are given on Γ_3 , while we still have Dirichlet data g_D on Γ_1 , the problem is of the form,

$$\int_{\Omega_1} \nabla u_h \cdot \nabla v_h \, dx = \int_{\Omega_1} f v_h \, dx + \int_{\Gamma_3} g_N v_h \, ds, \quad \forall v_h \in S^h \cap H_0^1(\Omega_1, \Gamma_1),$$

$$u_h|_{\Gamma_1} = g_D, \quad \gamma_0 u_h = g_D.$$

Here $\gamma_0 u_h$ is the trace on Γ_1 and $H_0^1(\Omega_1, \Gamma_1)$ is the subspace of $H^1(\Omega_1)$ with vanishing trace on Γ_1 . In this case x_3 is a vector of unknowns and the linear system is

$$\begin{pmatrix} K_{11} & K_{13} \\ K_{13}^T & K_{33}^{(1)} \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_3 \end{pmatrix}. \quad (2.4)$$

We note that the vector b_1 vanishes if f and g_D are zero in which case the vector b_3 represents the Neumann data on Γ_3 .

The stiffness matrix of the entire problem is of the form,

$$K = \begin{pmatrix} K_{11} & 0 & K_{13} \\ 0 & K_{22} & K_{23} \\ K_{13}^T & K_{23} & K_{33} \end{pmatrix}$$

where, by (2.1) and a simple computation,

$$K_{33} = K_{33}^{(1)} + K_{33}^{(2)}. \quad (2.5)$$

We note that the degrees of freedom have been partitioned into three sets of which the third, the separator set, corresponds to the nodes of Γ_3 . From the point of view of graph theory, the undirected graph of K becomes disconnected into two components if the nodes of the separator set and their incident edges are removed. If conforming finite elements are used, then it also follows from the assumptions on $a_\Omega(u, v)$ that K is positive definite, symmetric and as a consequence so are

K_{11} , K_{22} , and K_{33} .

We consider the linear system of algebraic equations of the form

$$Kx = \begin{pmatrix} K_{11} & 0 & K_{13} \\ 0 & K_{22} & K_{23} \\ K_{13}^T & K_{23} & K_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}. \quad (2.6)$$

By using block-Gaussian elimination, we can reduce this system to the positive definite, symmetric system,

$$\begin{aligned} Sx_3 &= (K_{33} - K_{13}^T K_{11}^{-1} K_{13} - K_{23}^T K_{22}^{-1} K_{23})x_3 \\ &= b_3 - K_{13}^T K_{11}^{-1} b_1 - K_{23}^T K_{22}^{-1} b_2 = \bar{b}_3. \end{aligned} \quad (2.7)$$

It is common practice to complete the process by solving (2.7) by a direct method.

The right hand side \bar{b}_3 can be obtained at the expense of solving the two subproblems on Ω_1 and Ω_2 , multiplying the resulting vectors by the sparse matrices K_{13}^T and K_{23}^T respectively and by subtracting the resulting vectors from b_3 . From now on, we will consider only the case when b_1 and b_2 are zero. Such a reduction can of course easily be accomplished.

The matrix S , is a so-called Schur complement; see Cottle [7], and can be expensive to compute and store. However, we notice that Sy can be computed for a given vector y at the expense of solving the two subproblems with the sparse right hand sides $K_{13}y$ and $K_{23}y$, respectively, and certain sparse matrix and vector operations. In the next section, we will develop iterative methods which only require S in terms of such matrix-vector products.

The cost of computing Sy depends primarily on the efficiency of the solvers for the subproblems. It should also be noted that if a Gaussian elimination method is used, advantage can be taken of the sparsity of the vectors $K_{13}y$ and $K_{23}y$.

Thus when the lower triangular systems of equations are solved, the computation can begin with the first equation which has a nonzero right hand side. Similarly, the solution of the upper triangular systems can be stopped as soon as all the components of $K_{11}^{-1}K_{13}y$ and $K_{22}^{-1}K_{23}y$, necessary for computing $K_{13}^T(K_{11}^{-1}K_{13}y)$ and $K_{23}^T(K_{22}^{-1}K_{23}y)$, have been found. This can effectively reduce the size of the triangular systems necessary to carry out the iteration steps. It is thus particularly advantageous if all the variables at nodes adjacent to Γ_3 are ordered late. It should be noted, however, that such a constraint may be hard to impose on existing software or that it may lead to an increase in the time required to factor K_{11} and K_{22} into their triangular factors.

We will need the Schur complements with respect to the matrices $K^{(1)}$ and $K^{(2)}$ defined in (2.2). They are,

$$S^{(1)} = K_{33}^{(1)} - K_{13}^{(1)T} K_{11}^{(1)-1} K_{13}^{(1)} \quad \text{and} \quad S^{(2)} = K_{33}^{(2)} - K_{23}^{(2)T} K_{22}^{(2)-1} K_{23}^{(2)}, \quad (2.8)$$

Using (2.7) and (2.8), we find that

$$S = S^{(1)} + S^{(2)}. \quad (2.9)$$

The mappings S and $S^{(1)}$ play an important role in what follows. The vector $S^{(1)}y$ can be computed by solving a Dirichlet problem (2.3) and then applying the matrix of (2.4), which corresponds to a Neumann case, to the solution vector. This can be seen by a straightforward computation:

$$\begin{pmatrix} K_{11} & K_{13} \\ K_{13}^T & K_{33}^{(1)} \end{pmatrix} \begin{pmatrix} K_{11} & K_{13} \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ S^{(1)}y \end{pmatrix}.$$

From our previous discussion we see that this is a Dirichlet-Neumann map which takes the Dirichlet data y on Γ_3 into the discrete Neumann data $S^{(1)}y$ on the same set. For a discussion of the continuous case, see next section.

CONJUGATE GRADIENT ALGORITHMS FOR SUBSTRUCTURED PROBLEMS AND AN INFORMAL THEORY.

The general theory of conjugate methods is quite well known and it will therefore be discussed only very briefly; see Concus, Golub and O'Leary [6], Hestenes [14] or Luenberger [16].

Let $Ax = b$ be a linear system of algebraic equations with a positive definite, symmetric matrix A . Let $x^{(0)}$ be an initial guess and $r^{(0)} = b - Ax^{(0)}$ the initial residual. The k -th iterate in the standard conjugate gradient method, $x^{(k)}$, can then be characterized as the minimizing element for the variational problem

$$\min_{y-x^{(0)}} \left(\frac{1}{2} \right) y^T A y - y^T b$$

where $y-x^{(0)}$ varies in the linear space spanned by $r^{(0)}, Ar^{(0)}, \dots, A^{k-1}r^{(0)}$. By expanding in eigenvectors of A , it can be established that

$$(x^{(k)} - x)^T A(x^{(k)} - x) / (x^{(0)} - x)^T A(x^{(0)} - x)$$

is bounded from above by,

$$\min_{p \in P_{k-1}} \max_{\lambda \in \sigma(A)} (1 - \lambda p(\lambda))^2 \quad (3.1)$$

See Luenberger [16]. Here x is the exact solution, P_{k-1} the space of all polynomial of degree $k-1$ and $\sigma(A)$ the spectrum of A . This bound can be used to establish that the convergence is rapid if A is well conditioned and that the rate of convergence can be bounded uniformly for entire families of operators if all the eigenvalues fall in a fixed interval.

Preconditioned conjugate gradient methods have been studied extensively in recent years. The idea goes back to the mid-fifties; see Hestenes [14]. Let A_0 be another positive definite, symmetric operator for which it is feasible to solve auxiliary systems of the form $A_0 y = c$ repeatedly for different right hand sides. In one of the versions of the method, the original problem $Ax = b$ is transformed into $AA_0^{-1}y = b$. The iterate $y^{(k)}$ is sought as the sum of an initial guess $y^{(0)}$ and a linear combination of $r^{(0)}, AA_0^{-1}r^{(0)}, \dots, (AA_0^{-1})^{k-1}r^{(0)}$. If an appropriate inner product is used, a convenient recursion formula results, see e.g. Proskurowski and Widlund [21]. Each step of this algorithm requires the solution of an auxiliary linear system. It is important to note that the estimate (3.1) still holds, but that now the eigenvalues of AA_0^{-1} , i.e. those of the symmetric generalized eigenvalue problem $A\phi = \lambda A_0\phi$, are of relevance rather than those of A . It is also worth noting that the estimate (3.1) can be used to show particularly rapid convergence if the eigenvalues are clustered.

For the problem at hand, we first consider the solution of equation (2.7) without preconditioning. From (2.9), we see that $Sy = S^{(1)}y + S^{(2)}y$. It is therefore at least plausible that S will be ill conditioned if $S^{(1)}$ and $S^{(2)}$ are. As shown in section 2, $S^{(1)}$ represents a Dirichlet-Neumann map and therefore involves a loss of a derivative in $L_2(\Gamma_3)$. Such a map will have a spectral condition number proportional to the number of nodes on Γ_3 . In order to clarify this point, we consider the continuous case, leaving the details on the finite element case to the forthcoming paper by Björstad and Widlund [4].

Thus consider two harmonic functions u_1 and u_2 defined on Ω_1 and Ω_2 respectively. These functions vanish on Γ_1 and Γ_2 and have the same trace on Γ_3 . They can therefore be combined to form $u \in H_0^1(\Omega)$. This function satisfies

$$a_{\Omega}(u, v) = f(v), \quad \forall v \in H_0^1(\Omega), \quad (3.2)$$

where the linear functional f has its support on Γ_3 . It is easy to show that

$$a_{\Omega}(u, v) = \int_{\Omega_1} \nabla u_1 \cdot \nabla v \, dx + \int_{\Omega_2} \nabla u_2 \cdot \nabla v \, dx = \int_{\Gamma_3} \frac{\partial u}{\partial \nu} v \, ds - \int_{\Gamma_3} \frac{\partial u}{\partial \nu} v \, ds = \int_{\Gamma_3} \left[\frac{\partial u}{\partial \nu} \right] v \, ds,$$

where ν is the normal outward with respect to Ω_1 . We can therefore rewrite equation (3.2) as

$$\left[\frac{\partial u}{\partial \nu} \right] = f,$$

where $\left[\frac{\partial u}{\partial \nu} \right]$ corresponds to Sy . Following Lions and Magenes [15] we see that for $u \in H_0^1(\Omega)$,

$$\gamma_0 u \in H_{00}^{1/2}(\Gamma_3) = \{u \in H_0^1(\Gamma_3); \rho^{-1/2} u\}_{L^2(\Gamma_3)} < \infty\},$$

where ρ is the distance of a point on Γ_3 to its end points. It then follows, from a standard variational argument, that $\frac{\partial u}{\partial \nu}$, $\frac{\partial u}{\partial \nu}$ and $\left[\frac{\partial u}{\partial \nu} \right]$ belong to the dual space of $H_{00}^{1/2}(\Gamma_3)$; i.e. a derivative is lost in comparison with $\gamma_0^3 u$.

In view of what we have just learned, it is natural to try to find a preconditioner which also involves the loss of a derivative in $L_2(\Gamma_3)$. A natural choice would be a tangential derivative but that is not a symmetric operator. Instead we can use the square root of the negative of a discretization of the Laplacian on Γ_3 . Such a method is practical for problems in the plane and has been tested; see section 4. We denote this operator as S .

An even better method involves the solution of a system,

$$\begin{bmatrix} K_{11} & 0 & K_{13} \\ 0 & K_{22} & K_{23} \\ K_{13}^T & 0 & K_{33}^{(1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ y \end{bmatrix}.$$

This solution can be accomplished by solving equation (2.4) with the right hand side $(0, y)^T$ and then the discrete Dirichlet problem on Ω_2 , using x_3 as data. The relevant mapping is now $SS(1)^{-1}$ since

$$\begin{bmatrix} K_{11} & 0 & K_{13} \\ 0 & K_{22} & K_{23} \\ K_{13}^T & K_{23}^T & K_{33} \end{bmatrix} \begin{bmatrix} K_{11} & 0 & K_{13} \\ 0 & K_{22} & K_{23} \\ K_{13}^T & 0 & K_{33}^{(1)} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ SS(1)^{-1} y \end{bmatrix}.$$

It can be shown that this operator is uniformly well conditioned; see Björstad and Widlund [4].

We note that we need not construct any auxiliary operator when this preconditioner is used. It is also interesting to note that if a rectangular region is cut in half, and treated fully symmetrically, then $S = 2S(1)$ and the conjugate gradient iteration converges in one step.

NUMERICAL EXPERIMENTS WITH A SUBSTRUCTURED PROBLEM.

We will only give a brief report on a few of the many experiments which we have carried out for the five point approximation of Poisson's equation on regions which are unions of two rectangles. The choice of such regions greatly simplifies the experiments and makes it feasible to conduct many experiments with very many degrees of freedom since fast Poisson solvers can be used to solve the subproblems. We note that this simple finite difference approximation can be viewed as a conforming finite element approximation on a mesh of right triangles using piecewise linear functions; see Strang and Fix [22].

In the experiments considered here, we consider the union of two rectangles with corners at the points $(0,0)$, $(1,0)$, $(1,1/2)$, $(0,1/2)$ and $(1/8,1/2)$, $(5/8,1/2)$, $(5/8,1)$, $(1/8,1)$ respectively. The relevant parameters are the number of mesh points q on Γ_3 , the interval between $(1/8,1/2)$ and $(5/8,1/2)$, and the total number of degrees of freedom. We have used data which are consistent with an exact solution $u(x,y) = x^2 + y^2 - xe^{x \cos y}$. We have found no real difference between the performance of our method for this and other cases.

We first show, in Table I, how the number of iterations grows as a function of q using the operator $SS(1)^{-1}$. We stopped the iterations at the level of the truncation error. The initial guess was the zero function. We note that the overall number of degrees of freedom increases quadratically with q and equals 48641 for $q = 127$.

TABLE I

q	Number of Iterations	Maximum Error on Ω
3	2	3.66×10^{-4}
7	3	9.59×10^{-5}
15	3	2.45×10^{-5}
31	4	6.09×10^{-6}
63	4	1.49×10^{-6}
127	5	3.02×10^{-7}

In Table II, we compare this preconditioner, $S^{(1)}$ with J , which is the square root of the negative of the discrete Laplacian on Γ_3 , for the case when $q = 127$.

TABLE II

No. of Iterations	Preconditioner: $S^{(1)}$	Preconditioner: J
0	3.79×10^{-1}	3.79×10^{-1}
1	1.25×10^{-2}	3.22×10^{-2}
2	7.48×10^{-4}	4.01×10^{-3}
3	2.56×10^{-5}	5.26×10^{-4}
4	4.42×10^{-7}	8.74×10^{-5}
5	3.02×10^{-7}	1.05×10^{-5}
6	3.02×10^{-7}	1.33×10^{-6}
7	3.02×10^{-7}	3.08×10^{-7}
8	3.02×10^{-7}	3.03×10^{-7}

Ten to twelve iterations were required to decrease the maximum error by a factor of 10 if no preconditioner was used.

We also give some information on the spectrum of $SS^{(1)^{-1}}$ for $q = 63$. The smallest eigenvalues were found to be $\lambda_1 = 1.713$, $\lambda_2 = 1.777$, ..., $\lambda_5 = 1.992$ with all eigenvalues less than or equal to 2.000. The corresponding spectrum for SJ^{-1} shows a somewhat less pronounced cluster with $\lambda_1 = 1.733$, $\lambda_2 = 1.804$, ..., $\lambda_5 = 2.008$, ..., $\lambda_{63} = 2.828$.

DIRICHLET'S PROBLEM FOR THE BIHARMONIC EQUATION.

The biharmonic Dirichlet problem has the form,

$$\begin{aligned} \Delta^2 \psi &= f \text{ in } \Omega, \\ \gamma_0 \psi &= g_0, \\ \gamma_1 \psi &= g_1, \end{aligned} \quad (5.1)$$

where $\gamma_0 \psi$ and $\gamma_1 \psi$ are traces; i.e. the restrictions of ψ and $\partial \psi / \partial \nu$ to the boundary. For a discussion of the existence of traces, a definition of the Sobolev spaces used in this section and a general introduction to elliptic problems; see Lions and Magenes [15]. We assume that the region Ω is plane, bounded and simply connected and denote the boundary by Γ .

In our discussion, we will use the following regularity result; cf. Glowinski and Pironneau [11]:

Let Γ be sufficiently smooth, and assume that $f \in L^2(\Omega)$, $g_0 \in H^{3/2}(\Gamma)$ and $g_1 \in H^{1/2}(\Gamma)$. Then the solution of (5.1) satisfies $\psi \in H^2(\Omega)$ and $\gamma_0(\Delta \psi) \in H^{-1/2}(\Gamma)$.

It is easy to show that we only need to consider the case where $f = 0$ and $g_0 = 0$ in what follows. Following Glowinski and Pironneau [11], we attempt to solve equation (5.1) by finding λ , the trace of the vorticity $\omega = -\Delta \psi$, for which $\gamma_1 \psi = g_1$. The relation between $\gamma_0 \omega$ and $\gamma_1 \psi$ is given by the system,

$$\begin{aligned} -\Delta \omega &= 0 \text{ in } \Omega, \\ \gamma_0 \omega &= \lambda, \\ -\Delta \psi &= \omega \text{ in } \Omega, \\ \gamma_0 \psi &= 0, \end{aligned} \quad (5.2)$$

which can be shown to define a selfadjoint, $H^{-1/2}(\Gamma)$ -elliptic operator A which maps $H^{-1/2}(\Gamma)$ onto $H^{1/2}(\Gamma)$. At the expense of solving two Dirichlet problems for the simplest second order elliptic problem, we can thus compute $A\lambda$ for a given λ and can then use the conjugate gradient method to find λ and the solution of problem (5.1).

There have been many attempts to use related methods for finite difference approximations of (5.1); see Björstad [1,2] and Glowinski and Pironneau [11] for a discussion and references. The idea is systematically developed by Glowinski and Pironneau in the framework of mixed finite elements. These have proven to be the most successful class of methods for solving problem (5.1). We review some of their work stressing the similarity with the work discussed in sections 2 and 3.

The mixed finite element methods are defined by,

$$\int_{\Omega} \nabla \omega_h \cdot \nabla v_h \, dx = 0, \quad \forall v_h \in S_0^h, \quad \omega_h \in S^h,$$

$$\int_{\Omega} \nabla \psi_h \cdot \nabla v_h \, dx = \int_{\Omega} \omega_h v_h \, dx, \quad \forall v_h \in S_0^h, \quad \psi_h \in S_0^h,$$

and

$$\int_{\Omega} \nabla \psi_h \cdot \nabla u_h \, dx = \int_{\Omega} \omega_h u_h \, dx + \int_{\Gamma} g_1 u_h \, ds, \quad \forall u_h \in M_h.$$

Here $S^h \subset H^1(\Omega)$ is the finite element subspace, $S_0^h = S^h \cap H_0^1(\Omega)$ and M_h the space spanned by the basis functions associated with the modes on Γ . M_h is thus a subspace of S^h which is complementary to S_0^h . With ψ_1 denoting the vector of nodal values of ψ_h in Ω , ω_1 the vector of the negative of the nodal values of ω_h in Ω and ω_2 the vector values of $-\omega_h$ on Γ , we obtain a linear system of equations with a symmetric, indefinite coefficient matrix,

$$\begin{pmatrix} 0 & K_{11} & K_{12} \\ K_{11} & M_{11} & M_{12} \\ K_{12} & M_{12} & M_{22} \end{pmatrix} \begin{pmatrix} -\psi_1 \\ \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ b_2 \end{pmatrix}. \quad (5.3)$$

Here K_{11} is the stiffness submatrix which represents the couplings between pairs of nodes in Ω while K_{12} represents couplings between Ω and Γ . The matrices M_{11} , M_{12} and M_{22} are mass submatrices with elements $\int_{\Omega} \phi_i \phi_j \, dx$. The matrix K_{11} and the mass matrix are positive definite, symmetric. Diagonal subsystems associated with the vectors ψ_1 and ω_1 can be solved at the expense of two discrete problems for the Laplacian. As in the continuous case, we can reduce the problem (5.3) to a linear system for the trace of the vorticity. Using block Gaussian elimination, we can regard the resulting matrix, A_h , as a Schur complement. The symmetry of A_h follows by inspection. The fact that it is positive definite can be shown by using a result given in Cottle [7]. He uses Sylvester's theorem to relate the inertia of a block matrix with that of a principal minor and the corresponding Schur complement.

The matrix A_h is an approximation of the operator A . Glowinski and Pironneau used techniques from mathematical analysis to prove that in the case of a convex region, the condition number of A_h grows linearly with the number of nodes on Γ . This is not surprising since A maps $H^{-1/2}(\Gamma)$ into $H^{1/2}(\Gamma)$, i.e. a derivative is gained. They therefore suggested a preconditioner, based on the square root of a discretization of the operator $-(d^2/ds^2) + 1$. This provides an optimal method in the sense that the rate of convergence becomes independent of the number of degrees of freedom of the discrete model. This preconditioner is of course basically the same as the inverse of the operator J described in section 3.

In view of the success of the preconditioner $S^{(1)}$ for the substructured problems, we have initiated numerical experiments which will use the Dirichlet-Neumann map as a preconditioner for A_h . Given a vector y , $S^{(1)}y$ can be computed by solving the same discrete Laplace problem which is required for the computation of $A_h y$. Since the continuous as well as the discrete Dirichlet-Neumann map has a one dimensional null space of constants in this case, the operator $S^{(1)}$ will be modified by adding $ce e^T$, where $e^T = (1, 1, \dots, 1)$ and $c > 0$.

Fast algorithms for the biharmonic problem on rectangles were considered in the 1980 Stanford dissertation of Peter Bjørstad [1]; see also Bjørstad [2]. Although the region is often special such algorithms are nevertheless very useful in a

number of applications. The discrete model is the standard 13-point difference approximation of problem (5.1) on uniform meshes, but to simplify the notations we will instead describe Björstad's algorithm for the continuous case.

The task is thus the solution of equation (5.1) with a rectangular region Ω . The boundary $\Gamma = \Gamma_H \cup \Gamma_V$, where Γ_H is the union of the two horizontal sides. Because of the simple geometry there are alternatives to equation (5.2) when we search for problems which are easy to solve. Thus we can solve,

$$\begin{aligned} \Delta^2 \psi &= 0, \\ \gamma_0 \psi &= 0 \text{ on } \Gamma_H \cup \Gamma_V, \\ \gamma_1 \psi &= 0 \text{ on } \Gamma_H, \\ \gamma_0 \bar{\psi} &= 0 \text{ on } \Gamma_V, \end{aligned} \quad (5.4)$$

by separation of the variables. We do this by expanding in Fourier series in the horizontal direction and solving the resulting boundary value problems for a set of fourth order ordinary differential equations. We note that the data for equation (5.4) is quite special, but that we can always reduce our problem to this form by solving a preliminary separable problem. In the discrete case the appropriate algorithmic tools are the fast Fourier transform and the Choleski algorithm for five diagonal linear systems of equations.

Equation (5.4) can be used to define a mapping from $\gamma_0 \bar{\psi}$ to $\gamma_1 \psi$ on Γ_V . To find a preconditioner, we consider another separable problem for which we can use a Fourier expansion in the vertical direction.

$$\begin{aligned} \Delta^2 \bar{\psi} &= 0, \\ \gamma_0 \bar{\psi} &= 0 \text{ on } \Gamma_H \cup \Gamma_V, \\ \gamma_0 \bar{\omega} &= 0 \text{ on } \Gamma_H, \\ \gamma_1 \bar{\psi} &= 0 \text{ on } \Gamma_V. \end{aligned} \quad (5.5)$$

By evaluating $\gamma_0 \bar{\omega}$ on Γ_V for the solution of (5.5), we define the preconditioner which has proven to be highly successful in Björstad's work. A tight uniform bound on the spectrum of the preconditioned operator is rigorously established in his work. In computational practice, the error decreases by a factor of about 10 in each iteration step. We also note that advantage can be taken of the sparsity of data of the problem and the fact that in the iterations, the solution is only required on and close to Γ_V .

FINITE ELEMENT-CAPACITANCE MATRIX METHODS

There are occasions when a linear system of equations can be imbedded in a larger system which is easier to solve. The larger problem can for example be a discrete elliptic problem on a region for which a fast Poisson solver can be used. These techniques are also of interest if a series of problems, e.g. the same elliptic equation on different regions, can be imbedded in the same larger problem. In such a case, possibly extensive preprocessing of the larger problem might pay off if a sufficient number of smaller problems are to be solved. There are a number of different versions of these so called capacitance matrix methods. Here we will only consider iterative variants; see Dryja [9], O'Leary and Widlund [17], and Proskurowski and Widlund [20,21]. We also limit our discussion to self adjoint second order elliptic problems defined on a region which is imbedded in a larger region.

We can then adopt the same notations as in section 2, and regard Γ_3 as the boundary of Ω_1 , which is the region of interest. The complement of $\Omega_1 \cup \Gamma_3$ with respect to the larger region is denoted by Ω_2 . The vectors x_1 , x_2 and x_3 are associated with the nodal values in Ω_1 , Ω_2 and Γ_3 respectively. To simplify our arguments, we assume that all of the problems have symmetric, positive definite coefficient matrices.

For a Neumann problem on Ω_1 , we then obtain,

$$\begin{pmatrix} K_{11} & K_{13} \\ K_{13}^T & K_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_3 \end{pmatrix} \quad (6.1)$$

while the larger problem has the form

$$\begin{pmatrix} K_{11} & 0 & K_{13} \\ 0 & K_{22} & K_{23} \\ K_{13}^T & K_{23}^T & K_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \quad (6.2)$$

The value of b_2 is irrelevant and we can also, at the expense of one solution of equation (6.2), reduce the right hand side of (6.1) to the case when $b_1 = 0$. Since we can solve (6.2) in each iteration step, we can use the operator S as a preconditioner. Given $b_3 = y$ and with b_1 and b_2 zero, we can compute the solution of equation (6.2) and substitute x_1 and x_3 into equation (6.1). We then obtain the residual $b_3 - S^{(1)}S^{-1}y$. We know from section 3 that this operator has good spectral properties.

A Dirichlet problem on Ω_1 can be reduced to a Neumann problem on Ω_2 by the following device. The finite element Dirichlet problem has the form,

$$\begin{pmatrix} K_{11} & K_{13} \\ 0 & I \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_3 \end{pmatrix}$$

The same solution can also be obtained from,

$$\begin{pmatrix} K_{11} & 0 & K_{13} \\ 0 & K_{22} & K_{23} \\ 0 & K_{23}^T & K_{33}^{(2)} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ K_{23} b_3 \\ K_{33}^{(2)} b_3 \end{pmatrix} \quad (6.3)$$

which has the unique solution $(x_1, 0, b_3)^T$. Equation (6.3) can now be solved by the same technique as the previous problem on Ω_1 since it essentially is a Neumann problem on Ω_2 . The resulting iteration matrix is $S^{(2)}S^{-1}$.

It is also of interest to allow more general coefficient matrices in equation (6.2). These issues will be discussed in a forthcoming paper, see Widlund [23].

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ACKNOWLEDGEMENTS

This work was supported by the National Science Foundation under Contract NSF-MCS-8203236 and by the U. S. Department of Energy under contract DE-AC02-76-ERD3077-V at the Courant Mathematics and Computing Laboratory.

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